Tutorial notes-1 whizkiddong@gmail.com

Mean value thm:

Rolle $\xrightarrow{\text{deduce}}$ Lagrange (single f)

↑ special case

Rolle $\xrightarrow{\text{deduce}}$ Cauchy. (consider 2 functions: f, g)

conditions of Rolle thm:

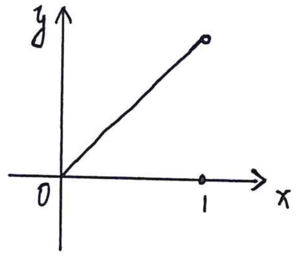
(1) f is continuous in $[a,b]$ (closed interval)

(2) f is differentiable in (a,b) (open interval)

(3) $f(a) = f(b)$

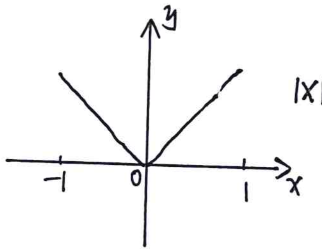
~~both~~ all 3 conditions are necessary, like:

(1)



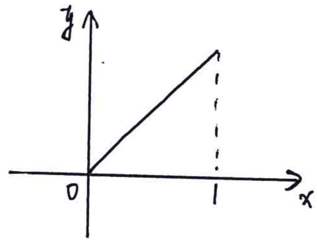
$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & x = 1 \end{cases}$$

(2)



$|x|$

(3)



$f(x) = x$

Q1. Assume $\frac{a_0}{n+1} + \cdots + a_n = 0$, try to prove that:

$a_0 x^n + \cdots + a_n = 0$ has at least one root in $(0,1)$

Pf: consider $f(x) = \frac{a_0}{n+1} x^{n+1} + \cdots + a_n x$

the idea comes from we need the form like $\frac{a_0}{n+1}$ to apply our condition.

so $f(0) = 0$, and $f(1) = \frac{a_0}{n+1} + \cdots + a_n = 0$ from condition.

so Rolle thm satisfied. there is a $\xi \in (0,1)$ s.t $f'(\xi) = 0$

which is $f'(\xi) = a_0 \xi^n + a_1 \xi^{n-1} + \cdots + a_n = 0$, so ξ is a root.

but we don't know what it is.

Q2. Assume $f(x)$ is continuous in $[0,1]$, differentiable in $(0,1)$,

And $|f'(x)|<1$, $f(0)=f(1)$, try to show: $\forall\, x_1, x_2 \in (0,1)$, $|f(x_1)-f(x_2)|<1$.

Pf: ①. If $|x_1-x_2|<\frac{1}{2}$, consider lagrange thm in $[x_1, x_2]$ (just assume $x_1<x_2$)

$$|f(x_1)-f(x_2)| = |f'(\xi)(x_1-x_2)| = |f'(\xi)|\cdot|x_1-x_2| < \frac{1}{2}.$$

② If $|x_1-x_2| \geq \frac{1}{2}$. consider 3 intervals $[0, x_1]$, $[x_1, x_2]$, $[x_2, 1]$:

$$\begin{aligned} |f(x_1)-f(x_2)| &= |f(x_1)-f(0)+f(1)-f(x_2)| \quad (f(0)=f(1)) \\ &\leq |f(x_1)-f(0)| + |f(1)-f(x_2)| \\ &= |f'(\xi_1)(x_1-0)| + |f'(\xi_2)(1-x_2)| \\ &< x_1 + 1 - x_2 \end{aligned}$$

For $|x_1-x_2| = x_2-x_1 \geq \frac{1}{2} \Rightarrow 1+x_1-x_2 = 1-(x_2-x_1) \leq \frac{1}{2}$ (assume $x_1<x_2$)

so $|f(x_1)-f(x_2)| < \frac{1}{2}$ in this case. too.

Q3. Assume $f(x)$ is continuous in $[a,b]$, differentiable in (a,b),

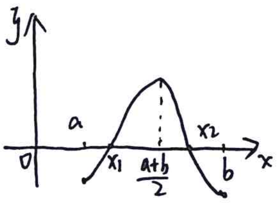
And: $f(a)\cdot f(b)>0$, $f(a)\cdot f(\frac{a+b}{2})<0$. Try to show:

For any $k\in R$, there exists $\xi\in(a,b)$ s.t $f'(\xi)=kf(\xi)$

Pf: From $f(a)\cdot f(b)>0$ we know $f(a)$, $f(b)$ should have the same ~~sign~~ sign, positive or negative.

so from $f(a)\cdot f(\frac{a+b}{2})<0$ means $f(a)$, $f(\frac{a+b}{2})$ have different sign.

The graph may like this:



so from the intermediate value thm of continuous function.

we know there are at least 2 roots of $f(x)$,

one $x_1\in(a, \frac{a+b}{2})$ and one $x_2\in(\frac{a+b}{2}, b)$

then consider $F(x)=e^{-kx}f(x)$ in $[x_1, x_2]$

$F(x_1)=e^{-kx_1}f(x_1)=0$ $(f(x_1)=0)$ $F(x_2)=e^{-kx_2}f(x_2)=0$ $(f(x_2)=0)$

Apply Rolle thm, there exists $\xi\in(x_1,x_2)\subset(a,b)$ s.t $F'(\xi)=e^{-k\xi}(f'(\xi)-kf(\xi))=0$

For $e^{-k\xi}\neq 0$. $\Rightarrow f'(\xi)-kf(\xi)=0 \Rightarrow f'(\xi)=kf(\xi)$

Remark: such method is called the auxiliary function, we try to construct a new function $F(x)$ that its derivative $F'(x)$ would satisfied the form of conclusion, then apply MVT in $F(x)$.

Q4. Assume $f(x)$ is twice differentiable in $(-\infty, +\infty)$, and $f(x)$ is bounded. Try to show: there exists $\xi \in (-\infty, +\infty)$ that $f''(\xi) = 0$.

Pf: we use the prove by contradiction.

Assume that there doesn't exist any ξ s.t $f''(\xi) = 0$ which means

$f''(x) > 0$ or $f''(x) < 0$ for all $x \in (-\infty, +\infty)$

(this is guaranteed by darboux thm which said the derivative would have intermediate property)

Now we consider $f''(x) > 0$ $x \in (-\infty, +\infty)$

$f''(x) > 0 \Rightarrow f'(x)$ is strictly increasing

take a fixed point x_0 which satisfied $f'(x_0) > 0$, consider $x > x_0$. then:

$$f(x) - f(x_0) = f'(\xi)(x - x_0) \quad (\text{Lagrange thm}) \quad \xi \in (x_0, x)$$
$$> f'(x_0)(x - x_0) \quad (\xi > x_0)$$

$$\Rightarrow f(x) > f(x_0) + f'(x_0)(x - x_0) \qquad (1)$$

For $f'(x_0) > 0$, so if we let $x \to +\infty$, the right-side of (1) goes to $+\infty$. so $f(x) \to +\infty$ which is a contradiction from $f(x)$ is bounded.

If $f'(x_0) < 0$, just consider $f(x) = f(x_0) + f'(\xi)(x - x_0)$ where $x < x_0$, so $\xi < x_0$

$$< f(x_0) + f'(x_0)(x - x_0) \quad \text{let } x \to -\infty, \text{ then } f(x) \to -\infty.$$

Similar to get contradiction when $f''(x) < 0$

So the assumption is wrong, there must exist some ξ s.t $f''(\xi) = 0$

Remark: the proof I give in the tutorial of Monday is wrong, the mistake is:

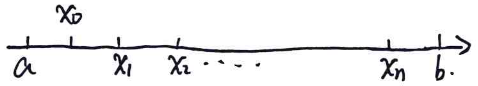
$$f(x)-f(x_0)=f'(\xi)(x-x_0) \Rightarrow |f'(\xi)| = \frac{|f(x)-f(x_0)|}{|x-x_0|} \leq \frac{2M}{|x-x_0|}$$

but this ξ actually should be $\xi=\xi(x)$ related to x, so when $x\to+\infty$, this ξ also changes, we can't sure that there are some ξ s.t $f'(\xi)=0$.

the counter-example is $y=f(x)=\arctan(x)$

$f'(x)=\frac{1}{1+x^2}\neq 0$, and $f''(x)=\frac{-2x}{(1+x^2)^2}$ would equal to 0 when $x=0$.

Q5. Assume $f(x)$ has $n+1$ different roots in $[a,b]$, and $f(x)$ has derivatives till to order n. try to show there is at least one root $\xi\in(a,b)$ s.t $f^{(n)}(\xi)=0$.

Pf: 

the $n+1$ roots are $x_0, \cdots x_n$

consider $[x_0,x_1]$. for $f(x_0)=f(x_1)=0$. use Rolle thm.

we get a $\xi_1^{(1)}\in(x_0,x_1)$, $f'(\xi_1^{(1)})=0$.

it's similar to consider $[x_k,x_{k+1}]$, $k=1,\cdots n-1$, so we can get n roots for $f'(x)$

$\xi_1^{(1)}, \xi_2^{(1)} \cdots\cdots \xi_n^{(1)} \rightarrow f'(\xi_i^{(1)})=0 \quad i=1,2\cdots n$.

repeat such process, the higher order derivative the less roots we have,

so finally when we consider $f^{(n)}(x)$, we can sure that there is at least one root $\xi\in(a,b)$, $f^{(n)}(\xi)=0$.

Math 1010C Term 1 2014
Supplementary exercises 3

The following exercises are not to be submitted, but they form an important part of the course, and you're advised to go through them carefully.

In supplementary exercise 2, we saw how one could find the absolute maximum / minimum of a continuous function on a closed and bounded interval. In the following, we will locate relative maximums / minimums of a function, and find the absolute maximum / minimum of a function on an unbounded interval (if it exists).

1. Find all critical points of the following functions on the indicated intervals. Determine whether these are relative maximums / minimums of the functions (they could be neither).
   (a) $f(x) = x^{1/3}(x-4)$, $\quad (-1, \infty)$
   (b) $g(x) = x\sqrt{8-x^2}$, $\quad (-2\sqrt{2}, 2\sqrt{2})$
   (c) $h(x) = x \ln x$, $\quad (0, \infty)$

2. For each of the following function,
   (i) Determine where the function is increasing, and where it is decreasing;
   (ii) Find all relative maximums / minimums of the function on $(-\infty, \infty)$;
   (iii) Determine whether any of these is an absolute extremum of the function on $(-\infty, \infty)$. (For this you will need to understand the behaviour of the function at $\pm\infty$.)
   (iv) Determine where the function is convex, and where it is concave;
   (v) Sketch the graph of the function.

   (a) $f(x) = x^3 - 12x - 5$
   (b) $g(x) = x^2(1-x^2)$
   (c) $h(x) = \dfrac{x}{x^2+1}$

3. For each of the functions and intervals in Question 1, determine whether the given function have an absolute maximum / minimum on the indicated intervals. (You'll have to understand the behaviour of these functions as x approaches the end-points of the intervals.) If yes, find the maximum / minimum values of the functions on the indicated intervals.

4. Determine whether the following functions have an absolute maximum / minimum on the indicated intervals. If yes, locate ALL points where the absolute maximum / minimum are achieved.
   (a) $f(x) = e^{2x} + e^{-x}$, $\quad [0, \infty)$
   (b) $g(x) = \dfrac{x^2-3}{x-2}$, $\quad (-\infty, 2)$
   (c) $h(x) = \dfrac{2x^2 - x^4}{x^4 - 2x^2 + 2}$, $\quad [-1, \infty)$

   (Credit: Many of the above functions are taken from Thomas' calculus, chapter 4.)

Supplementary Exercises 3.
From Math1010C homepage.
Included here for Reference
ONLY.

Math 1010C TUTORIAL - 7 23/10/2014 Jiang Lingyuan

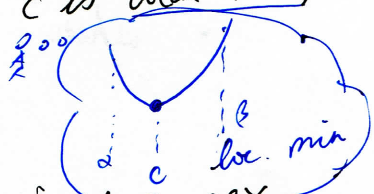
PLAN: (a). Supplementary exercise 3 ;
(b). L'Hospital's rule ;
(c). Q's on HW3 ;

(a) Supplementary exercise 3 (Math 1010 C webpage)

Recall: TOOLS !

- First derivative test : f cont. on (a,b) ; differentiable on $(a,b)\setminus\{c\}$

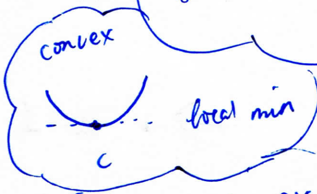
(i) $\begin{cases} f'(t) \le 0, & t \in (\alpha, c) \\ f'(t) \ge 0, & t \in (c, \beta) \end{cases}$ $\Rightarrow$ c is local min ;



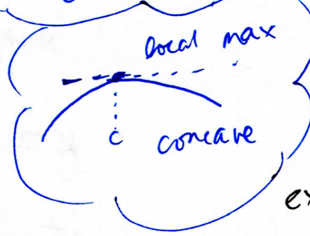
(ii) $\begin{cases} f'(t) \ge 0, & t \in (\alpha, c) \\ f'(t) \le 0, & t \in (c, \beta) \end{cases}$ $\Rightarrow$ c is local max

- Second derivative test : $f : (a,b) \to \mathbb{R}$ differentiable,
$f'(c) = 0$, & f' is differentiable at c ;

(i) $f''(c) > 0 \Rightarrow$ c is local min ;



(ii) $f''(c) < 0 \Rightarrow$ c is local max ;



Ex1. Find critical points, & determine whether relative extrema's :

(a) $f(x) = x^{1/3}(x-4)$. $x \in (-1, \infty)$,

(b) $g(x) = x\sqrt{8-x^2}$. $x \in (-2\sqrt{2}, 2\sqrt{2})$.

(c) $h(x) = x \ln x$; $x \in (0, \infty)$;

& Ex2 : whether these f'ns have absolute max / min on given interval.
& if yes, find extreme values ;

a) $f(x) = x^{1/3}(x-4)$, $\quad x \in (-1, \infty)$.

$$f'(x) = \frac{1}{3} \cdot x^{-\frac{2}{3}}(x-4) + x^{\frac{1}{3}} = x^{\frac{1}{3}}\left(\frac{x-4}{3x} + 1\right) = \frac{4(x-1)}{3x^{\frac{2}{3}}};$$

critical pt $\longleftrightarrow$ $f'(x) = 0$ $\longleftrightarrow$ $x = 1$; is the only critical pt;

observation:

x:	-1	0		1		4	∞
$f'(x)$:	$-$	∞	$-$	0	$+$		$+$
$f(x)$	5	0		-3		0	$+\infty$
				local min			

- when $x < 1$, $x \neq 0$, $\quad f'(x) < 0$, $\quad x > 1$, $\quad f'(x) > 0$ $\Big\}$ $\Rightarrow$ $x = 1$ is local min;

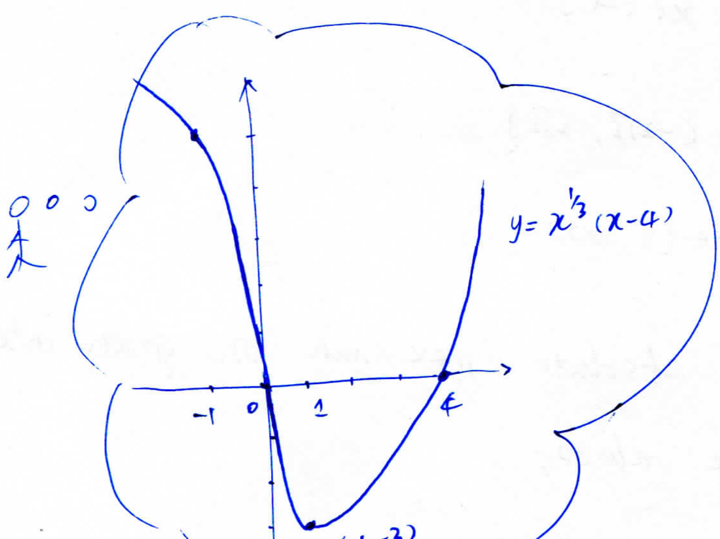
For Ex3: On $(-1, \infty)$:

$\Big\{$
- f is decreasing on $(-1, 1)$, & $f(0) = 0$ does not influence this
  $(f(-1) = 5,\ f(1) = -3)$ is cont. pt $\quad f'(0) = \infty$
- f is increasing on $(1, \infty)$;
  $(f(1) = -3,\ f(4) = 0,\ f(+\infty) = +\infty)$

$\Rightarrow$ • $x = 1$ is absolute min of f on $(-1, \infty)$, with min value $f(1) = -3$;

• f does not have an absolute max. on $(-1, \infty)$.
it is continuous on $(-1, +\infty)$, & $\lim_{t \to +\infty} f(t) = +\infty$;

$\square$



Ex1. (b) $g(x) = x\sqrt{8-x^2}$, $x \in (-2\sqrt{2}, 2\sqrt{2})$.

Sol'n $g'(x) = \sqrt{8-x^2} + x \cdot \dfrac{-2x}{2\sqrt{8-x^2}} = \dfrac{2(4-x^2)}{\sqrt{8-x^2}}$;

Then $g'(x) = 0 \iff x = \pm 2 \in (-2\sqrt{2}, 2\sqrt{2})$. Critical points;

Now observe that:

x:	$-2\sqrt{2}$		-2		0		2		$2\sqrt{2}$
$g'(x)$	$-\infty$	$-$	0	$+$			0	$-$	$-\infty$
$g(x)$	0		$\cancel{-8}$ -4		0		$\cancel{8}$ 4		0

Then since near $x=-2$, $\begin{cases} g'(x) < 0 & \text{for } x \in (-2\sqrt{2}, -2); \\ g'(x) > 0 & \text{for } x \in (-2, 2); \end{cases}$ $\Rightarrow$ $x=-2$ loc. min;

near $x = 2$, $\begin{cases} g'(x) > 0, & \text{for } x \in (-2, 2); \\ g'(x) < 0, & \text{for } x \in (2, 2\sqrt{2}); \end{cases}$ $\Rightarrow$ $x=2$ is local max;

※ for Ex1.

For Ex3, First observe $g(x)$ can be extended continuously on $[-2\sqrt{2}, 2\sqrt{2}]$ with $g(-2\sqrt{2}) = g(2\sqrt{2}) = 0$; $\rightsquigarrow$ (Can do what we do in Ex 2!)

& then according $g(x)$'s behaviour

x:	$-2\sqrt{2}$	$(-2\sqrt{2}, -2)$	-2	$(-2, 2)$	2	$(2, 2\sqrt{2})$	$2\sqrt{2}$
$g(x)'$:	$-\infty$	<0	0	>0	0	<0	$-\infty$
$g(x)$ behavior		$\searrow$	local min	$\nearrow$	local max	$\searrow$	
$g(x)$	0		-4 $\cancel{-8}$		$\cancel{8}$ 4		0

Compare $g(-2\sqrt{2}) = 0$, $g(-2) = \cancel{-8}$ -4, $g(2) = \cancel{8}$ 4, $g(2\sqrt{2}) = 0$;

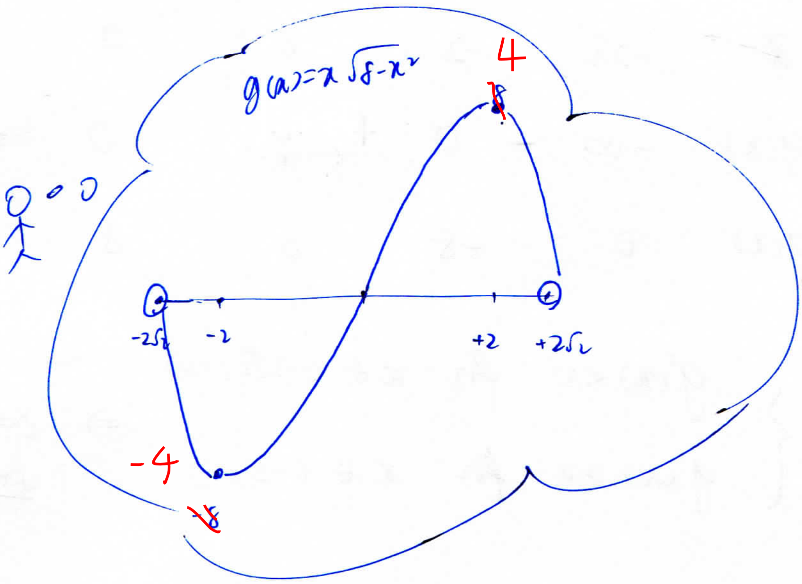
absolute ~~max~~ absolute max

<u>Conclusion</u>: $g(x)$ can achieve absolute max. & absolute min.

on $(-2\sqrt{2}, 2\sqrt{2})$, at $x=-2$ (abs. min.), w/ $g(-2) = \cancel{-8}\ -4$ abs. min value;

at $x=2$ (abs. max), w/ $g(2) = \cancel{8}\ 4$ abs. max value.

$\square$



(c). $h(x) = x\ln x$, $x \in (0, +\infty)$;

$h'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$;

Then $h'(x) = 0 \iff x = e^{-1} = \frac{1}{e} \approx 0.36788\ldots$

& $h(x)$ does not have absolute max:

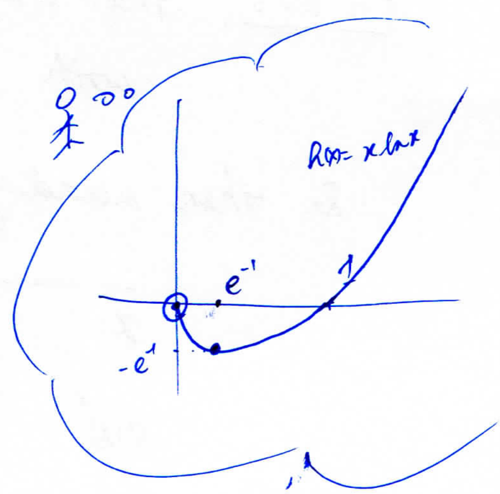
$\lim_{x\to+\infty} h(x) = +\infty$.

$\square$

& $h'(x) < 0 \iff x \in (0, \frac{1}{e})$

$h'(x) > 0 \iff x \in (\frac{1}{e}, +\infty)$;

$\Rightarrow$ $x = \frac{1}{e}$ is <u>local min</u>, ※ Ex. 1



For <u>Ex3</u>, observe that

	0	$(0, \frac{1}{e})$	$\frac{1}{e}$	$(\frac{1}{e}, +\infty)$	$+\infty$
			local min		
$h(x)$	0		$-\frac{1}{e}$		$+\infty$
$h'(x)$	$-\infty$	<0	0	>0	
		$\searrow$		$\nearrow$	

$\Rightarrow$ $x = \frac{1}{e}$ is <u>absolute min</u>, with $h(\frac{1}{e}) = -\frac{1}{e}$;

Ex2: (a) $f(x) = x^3 - 12x - 5$; where decreasing, increasing;

(b) $g(x) = x^2(1 - x^2)$; where convex, concave;

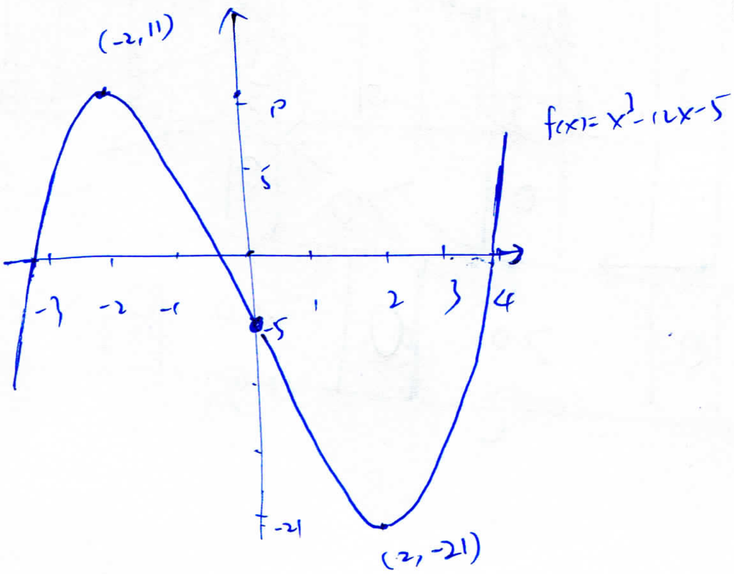
(c) $h(x) = \dfrac{x}{x^2+1}$; relative extreme, & whether absolute or not.

Sol'n: (a) $f(x) = x^3 - 12x - 5$;

$f'(x) = 3x^2 - 12 = 3(x-2)(x+2)$;

$f''(x) = 6x$;

x		x=-2		x=0		x=2	
f(x)		11		-5		-21	
f'(x)	>0	local max 0	<0		<0	local min 0	>0
f''(x)	<0 concave		<0 concave	0	convex >0		>0 convex



(b) $g(x) = x^2(1 - x^2)$,

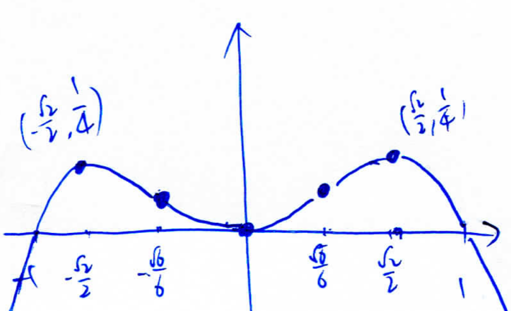
$g'(x) = 2x(1-x^2) + x^2 \cdot (-2x)$

$= 2x - 4x^3$

$= 2x(1 - 2x^2)$;

$g''(x) = 2(1 - 6x^2)$;

	$x=-\infty$	$x=-1$	$x=-\frac{\sqrt2}{2}$	$x=-\frac{\sqrt6}{6}$	$x=0$	$x=\frac{\sqrt6}{6}$	$x=\frac{\sqrt2}{2}$	$x=1$
g(x)	+∞	0	local max $\frac14$	$\frac{5}{36}$	local min 0	$\frac{5}{36}$	local max $\frac14$	0
g'(x)	>0	>0	0	<0	0	>0	0	<0
g''(x)	<0 concave	<0 concave	<0 concave	0	>0 convex	0	<0 concave	<0 concave

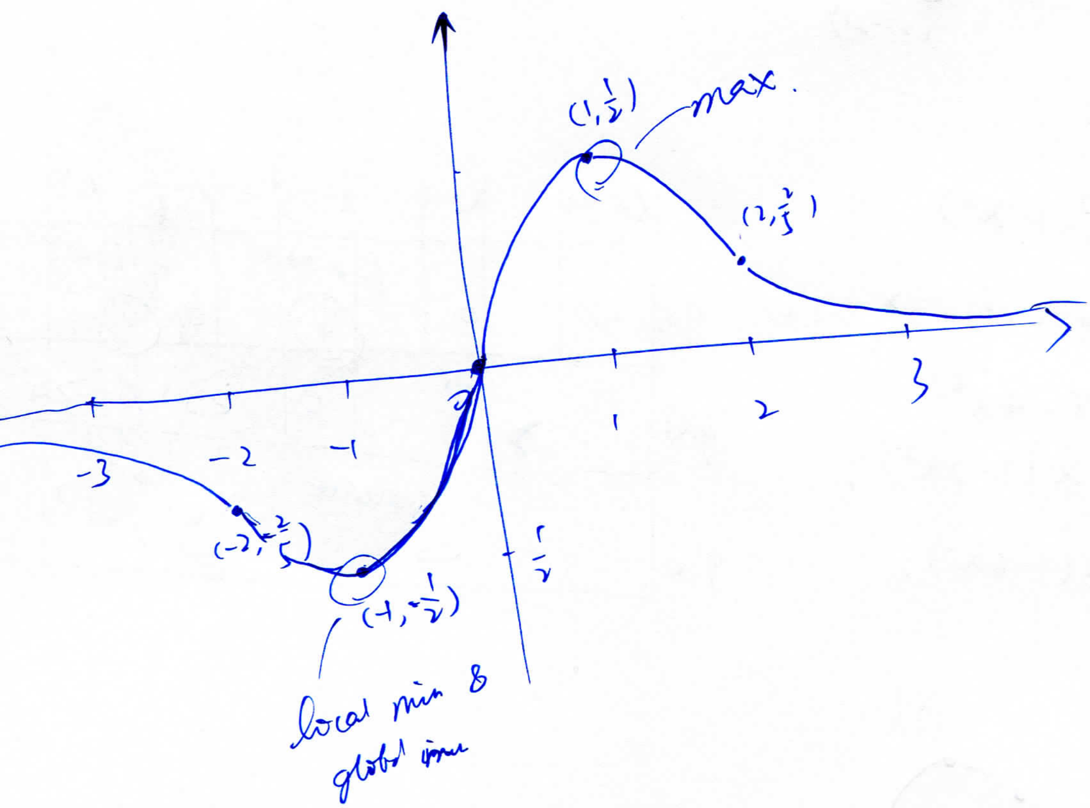


). $h(x) = \dfrac{x}{x^2+1}$;

$$h'(x) = \frac{x^2+1 - x(2x)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2} ;$$

$$h''(x) = \frac{1}{(x^2+1)^3} \cdot 2 \cdot x(x-2)(x+2) ;$$

x	$x=-\infty$		-3		-2		-1		0		1		2	3	$+\infty$
							local min				loc. max				
$h(x)$	0		$-\frac{3}{10}$		$-\frac{2}{5}$		$-\frac{1}{2}$		0		$\frac{1}{2}$		$\frac{2}{5}$	$\frac{3}{10}$	0
$h'(x)$		<0					0	>0			0	<0			
$h''(x)$		<0 concave			0	>0 convex			0	<0 concave			0	>0 convex	



□

B) L'Hôpital Rule

Thm 3.7 (P50, textbook) If f, g satisfies

$$\begin{cases} \cdot\ f(c) = g(c) = 0; \\ \cdot\ f, g \text{ are both differentiable on } (a,b); \text{ (except perhaps at } c) \\ \cdot\ g'(x) \neq 0, \text{ on } (a,b),\ \forall x \neq c; \\ \cdot\ \lim\limits_{x\to c} \dfrac{f'(x)}{g'(x)}\ \exists\ \&\ = L \text{ (finite number)}; \end{cases}$$

Then $\lim\limits_{x\to c} \dfrac{f(x)}{g(x)}\ \exists\ \&\ = L$;

Remark: More general forms:

$$\begin{cases} (1)\quad f(c) = g(c) = 0 \overset{\text{replace}}{\longleftrightarrow} \lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} g(x) = \pm\infty; \\ (2)\quad c \in (a,b) \overset{\text{replace}}{\longrightarrow} c = \pm\infty; \end{cases}$$

POINT: (i) L'Hôpital rule are most useful tool when dealing with evaluating limits involving indeterminate forms.

i.e. 7 forms: $\frac{0}{0}$; $\frac{\infty}{\infty}$; $0\cdot\infty$; $\infty - \infty$;

0^0, 1^∞ & ∞^0

$0^0 = e^{0\ln 0}$; $0\ln 0 = 0\cdot\infty = \frac{0}{0}$; $1^\infty = e^{\infty\ln 1}$; $\infty\ln 1 = \infty\cdot 0$; $\infty^0 = e^{0\cdot\ln\infty}$; $0\cdot\ln\infty = 0\cdot\infty$

& Have to check it is of one of above form before using L'Hôpital rule. (otherwise, clearly e.g. $\lim\limits_{x\to 0} \frac{\sin x}{\cos x} \neq \lim\limits_{x\to 0} \frac{+\cos x}{-\sin x}$)

I meant to say, It sufficient conditio
but not necessary
condition.

(ii) Sometimes L'Hôpital rule does not work, but it does not mean that the limit does not exist. E.g. $\lim\limits_{x\to\pm\infty} \dfrac{x+\sin x}{x+\cos x} = 1$, but L'Hôpital rule can not work here; (i.e it is sufficient but)

(iii) You can use L'Hôpital rule to verify the following useful equivalence relations easily. eg

$$x \to 0, \quad x \sim \sin x \sim \tan x \sim \arcsin x \sim \arctan x$$

$$\sim \ln(1+x) \sim \frac{a^x - 1}{\ln a} \sim \frac{(1+x)^\mu - 1}{\mu}; \quad (a>0, \mu \neq 0)$$

& $\frac{1}{2}x^2 \sim (1-\cos x)$; etc;

Example: . $\lim_{x\to 0} \frac{\sin x}{x}$; since $\frac{0}{0}$. $x \neq 0$ when $x \neq 0$, & $\lim_{x\to 0} \frac{\cos x}{1} = 1$

$$\Rightarrow \lim_{x\to 0} \frac{\sin x}{x} \ \exists \ \& \ \lim_{x\to 0} \frac{\sin x}{x} = 1;$$

. $\lim_{x\to 0} \frac{\ln(1+x)}{e^x - 1}$; since $\frac{0}{0}$, & $x \neq 0$, $e^x - 1 \neq 0$, &

$$\lim_{x\to 0} \frac{\ln(1+x)'}{(e^x-1)'} = \lim_{x\to 0} \frac{\frac{1}{1+x}}{e^x} = \lim_{x\to 0} \frac{1}{(1+x)e^x} = 1$$

$$\Rightarrow \lim_{x\to 0} \frac{\ln(1+x)}{e^x - 1} = 1;$$

etc;.

Examples (a) $\lim_{x\to 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right)$;

(b) $\lim_{x\to 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{1-\cos x}}$;

(c). $\lim_{x\to +\infty} \sqrt[x]{x} \left(= \lim_{x\to +\infty} x^{\frac{1}{x}} \right)$;

Sol'n: (a) $\lim_{x\to0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right)=\lim_{x\to0}\frac{x-\ln(1+x)}{x\ln(1+x)}$ $\left(\frac{0}{0}\right)$

TRY L'Hôpital

$$\lim_{x\to0}\frac{(x-\ln(1+x))'}{(x\ln(1+x))'}=\lim_{x\to0}\frac{1-\frac{1}{1+x}}{\ln(1+x)+\frac{x}{1+x}}=\lim_{x\to0}\frac{x}{(1+x)\ln(1+x)+x}\quad\left(\frac{0}{0}\right)$$

TRY L'Hôpital

$$\lim_{x\to0}\frac{(x)'}{[(1+x)\ln(1+x)+x]'}=\lim_{x\to0}\frac{1}{\ln(1+x)+2}=\frac{1}{2};$$

#

(b) $\lim_{x\to0}\underbrace{\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}}_{f(x)}$ TRY $\underline{\lim_{x\to0}\ln f(x)}=L$,

then $\lim_{x\to0}f(x)=e^L$;

$$\lim_{x\to0}\frac{1}{1-\cos x}\ln\frac{\sin x}{x}=\lim_{x\to0}\frac{\ln\frac{\sin x}{x}}{\frac{x^2}{2}}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0}\frac{\left(\ln\frac{\sin x}{x}\right)'}{\left(\frac{x^2}{2}\right)'}$$

$$=\lim_{x\to0}\frac{x\cos x-\sin x}{x^2\sin x}\overset{\left(\frac{0}{0}\right)}{=}\lim_{x\to0}\frac{(x\cos x-\sin x)'}{(x^3)'}=\lim_{x\to0}\frac{-x\sin x}{3x^2}$$

$=-\frac{1}{3}$. Hence $\lim_{x\to0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}=e^{-\frac{1}{3}}$;

#

(c) $\lim_{x\to+\infty}(x)^{\frac{1}{x}}$. still try

$$\lim_{x\to+\infty}\ln\sqrt[x]{x}=\lim_{x\to+\infty}\frac{\ln x}{x}\overset{\left(\frac{\infty}{\infty}\right)}{=}\lim_{x\to+\infty}\frac{\frac{1}{x}}{1}=0$$

$$\Rightarrow\lim_{x\to+\infty}\sqrt[x]{x}=1.$$

□

Tutorial 7

Topics : Rolle's, Lagrange's and Cauchy's mean value theorem.

Q1: Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable if $\lim_{x \to +\infty} f(x) = \lim_{x \to -\infty} f(x)$

Show that $\exists c \in \mathbb{R}$ s.t $f'(c) = 0$

Q2: Suppose $f : [a,b] \to \mathbb{R}$ is continuous

if $f^{(k)}(x)$ exists $\forall x \in (a,b), \forall k = 1, 2, \cdots, n$

and $f(a_i) = 0 \quad \forall i = 0, 1, \cdots, n$ where $a_0 < a_1 < \cdots < a_n$

Show that $\exists c \in (a,b)$ s.t $f^{(n)}(c) = 0$.

Q3: Suppose $f:[a,b] \to \mathbb{R}$ is continuous

if f is differentiable on (a,b) and $f'(x) = 0 \ \forall x \in (a,b)$

show that f is a constant function on $[a,b]$.

Q4: Suppose $f:(a,b) \longrightarrow \mathbb{R}$ is continuous

if f is differentiable at $(a,b) \setminus \{c\}$ and $\lim_{x \to c} f'(x)$ exists.

show that f is differentiable at $x = c$.

Recall:

Suppose $f, g : [a,b] \to \mathbb{R}$ is continuous
and f, g are differenticble on (a,b)

Rolle's MVT: If $f(a) = f(b)$ then $\exists c \in (a,b)$ s.t. $f'(c) = 0$.

Lagrange's MVT: There exist $c \in (a,b)$ s.t. $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

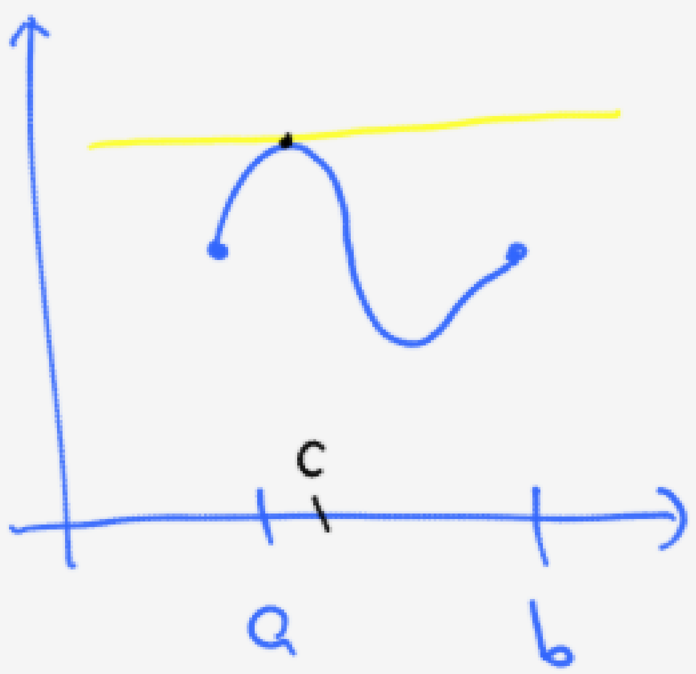
Cauchy's MVT: There exist $c \in (a,b)$ s.t. $[f(b) - f(a)]g'(c) = [g(b) - g(a)]f'(c)$

equivalently if $g'(c) \neq 0$, $g(b) - g(a) \neq 0$

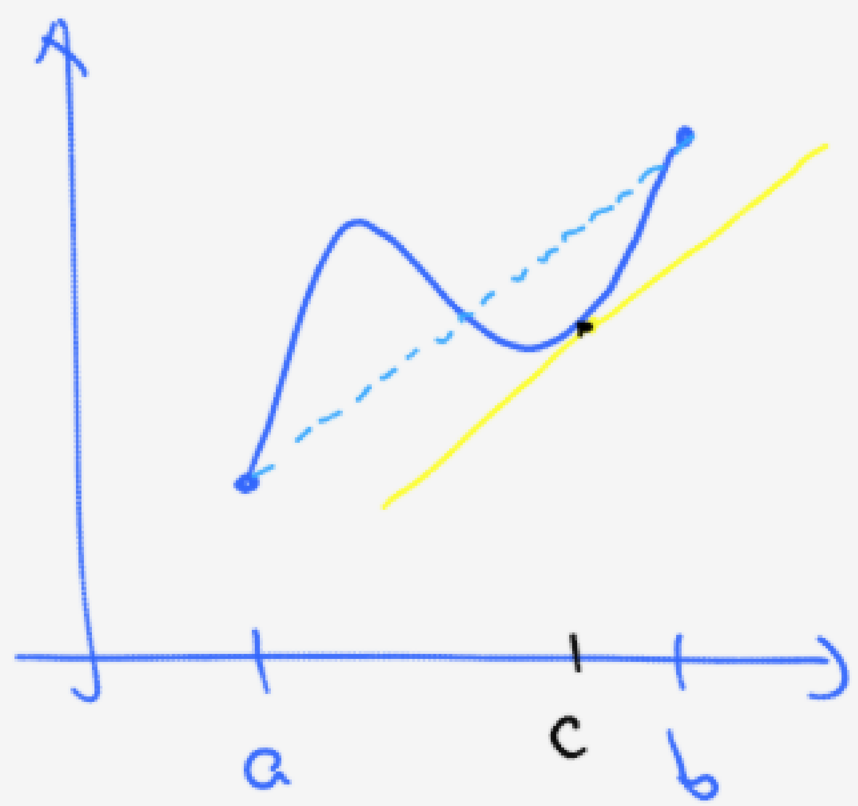
then $$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Geometrically

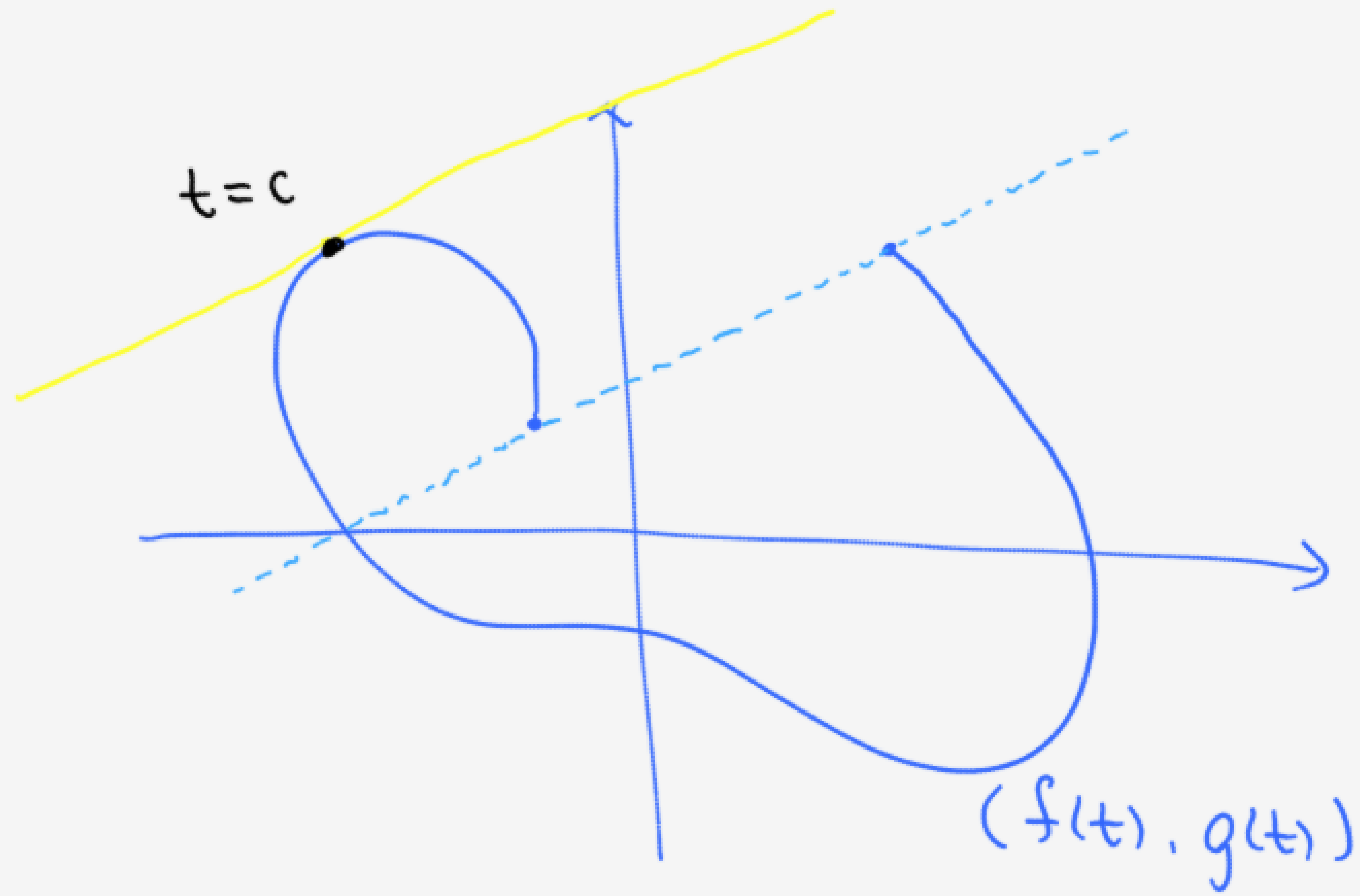
Rolle's



Lagrange's



Cauchy's



<u>Sol^n</u>

Q1) Consider $x=0$, we have $f(0) = \lim\limits_{x\to\pm\infty} f(x)$ or $f(0) \neq \lim\limits_{x\to\pm\infty} f(x)$

Case 1 : $f(0) \neq \lim\limits_{x\to\pm\infty} f(x)$

choose y_0 between $f(0), \lim\limits_{x\to\pm\infty} f(x)$

i.e. $\min\left\{ f(0), \lim\limits_{x\to\pm\infty} f(x) \right\} < y_0 < \max\left\{ f(0), \lim\limits_{x\to\pm\infty} f(x) \right\}$

since f is continuous on $\mathbb{R}$

$\exists$ $a \in (-\infty, 0)$ and $b \in (0, \infty)$

s.t. $f(a) = f(b) = \lim\limits_{x\to\pm\infty} f(x)$

By Rolle's thm $\exists$ $c \in (a,b)$ s.t. $f'(c) = 0$

Case ② if $f(0) = \lim_{x \to \pm\infty} f(x)$

consider $x = 1$

if $f(1) \neq \lim_{x \to \pm\infty} f(x)$ then repeat argument of Case ①.

if $f(1) = \lim_{x \to \pm\infty} f(x) = f(0)$ then by Rolle's thm

$\exists\ c \in (0,1)$ s.t. $f'(c) = 0$

Q2) Since $f(a_0) = f(a_1) = \cdots = f(a_n) = 0$

for $a_0 < a_1 < \cdots < a_n$

By Rolle's MVT $\exists\ b_i \in (a_i, a_{i+1})$, $i = 0, 1, \cdots, n-1$

s.t. $f'(b_0) = f'(b_1) = \cdots = f'(b_{n-1}) = 0$

where $b_0 < b_1 < \cdots < b_{n-1}$

Assume that for $k = 0, 1, \cdots, n-1$ s.t.

$f^{(k)}(x_0) = \cdots = f^{(k)}(x_{n-k}) = 0$

By Rolle's thm $\exists\ y_i \in (x_i, x_{i+1})$ $i = 0, \cdots, n-k-1$

s.t. $f^{(k+1)}(y_i) = f^{(k)\prime}(y_i) = 0$

Inductively, we have $f^{(n)}(y_i) = 0$, $i = 0 \Rightarrow f^{(n)}(c) = 0$ for $c = y_0$.

Q3) By Lagrange's MVT,

Let $x \in (a,b)$, $\frac{f(b)-f(x)}{b-x} = f'(c) \quad \exists\, c \in (x,b)$

$\Rightarrow \quad 0 = f'(c) = \frac{f(b)-f(x)}{b-x} \quad \Rightarrow \quad f(b)-f(x) = 0$

$\Rightarrow \quad f(x) = f(b) \quad \forall x \in (a,b)$

Hence f is constant on $(a,b]$. [need $f(a)=f(b)$].

By Rolle's thm, $\exists\, c \in (a,b)$ s.t. $0 = f'(c) = \frac{f(b)-f(a)}{b-a}$

$\Rightarrow \quad f(b) = f(a)$

Hence $f(x) = f(b) = f(a)$ on $x \in [a,b]$ //

Q4) Let $h \neq 0$, $\dfrac{f(c+h) - f(c)}{h} = \dfrac{f(x) - f(c)}{x - c}$ (by sub $x = c+h$)

$= f'(t_x)$ where $x \neq c$ and

$\min\{x, c\} < t_x < \max\{x, c\}$

As $h \to 0$, we have $x \to c$; thus $\lim\limits_{x \to c} t_x = c$

Hence

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = \lim_{x \to c} f'(t_x) = \lim_{t \to c} f'(t) \text{ exists}$$

(by assumption)

Hence f is differentiable at $x = c$

and $f'(c) = \lim\limits_{t \to c} f'(t)$

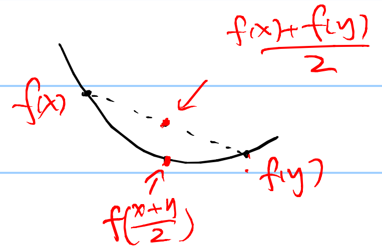
便笺标题 2014/10/20

Tutorial 7. jmin@math.cuhk.edu

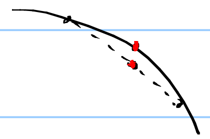
* Sketch graph of a function.

Second derivative ~ convex, concave.

Convex:



Concave:



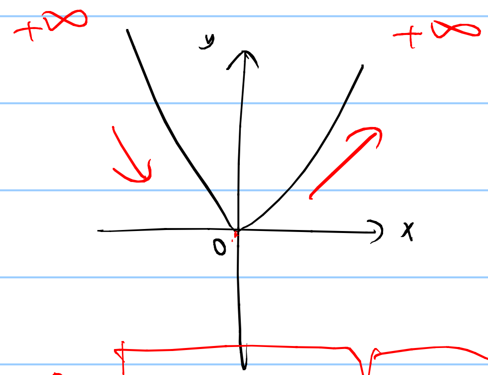
$$f(\tfrac{x+y}{2}) < \tfrac{1}{2}f(x)+\tfrac{1}{2}f(y) \qquad\qquad f(\tfrac{x+y}{2}) > \tfrac{1}{2}f(x)+\tfrac{1}{2}f(y)$$

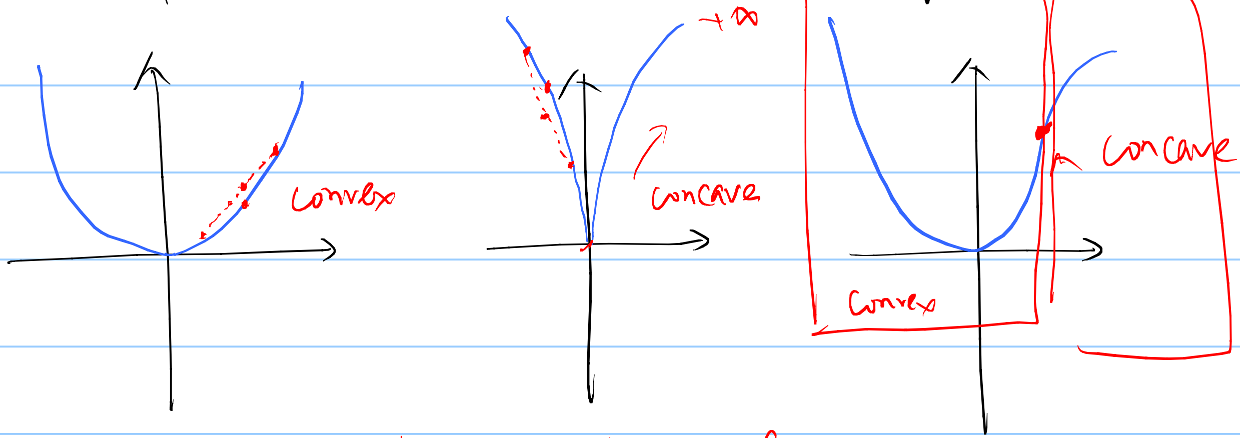
$$\Uparrow \qquad\qquad\qquad\qquad \Uparrow$$

$$f''(x) > 0 \qquad\qquad\qquad\qquad f''(x) < 0$$

Eg. graph of $y = x^2$.

Zeros. positive/negative
asymptotics
first derivative
$\Rightarrow$





Cannot distinguish these by above information

$\Downarrow$

Need second derivative, $y'' = 2 > 0$

So the function is always convex

* Mean Value Thm :

Condition : Remember to check this condition when you want to apply MVT !!!

* $f(x)$ is cont's on $[a,b]$ and differentiable on (a,b)

Rolle's thm	<	Lagrange's MVT	<	Cauchy's MVT
$f(a)=f(b)$ $\Downarrow$ $\exists\, \xi\in(a,b)$ s.t. $f'(\xi)=0$		$\exists\, \xi \in (a,b)$ s.t. $f'(\xi)=\frac{f(b)-f(a)}{b-a}$		$g'(x)\neq 0$ for $x\in(a,b)$ $\Downarrow$ $\exists\, \xi\in(a,b)$ s.t. $\frac{f'(\xi)}{g'(\xi)}=\frac{f(b)-f(a)}{g(b)-g(a)}$

MVT is used to do proofs :

Eg. $a<c<b$, $f:(a,b)\to \mathbb{R}$ is cont's. f is differentiable on $(a,b)\setminus\{c\}$, and $\lim_{x\to c} f'(x)$ exists.

Prove that f is differentiable at c.

and $f'(c)=\lim_{x\to c} f'(x)$.

Pf : To prove differentiability, we do it using definition

f is differentiable at c if

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \text{ exists}$$

Now $\frac{f(c+h)-f(c)}{h} = \frac{f(c+h)-f(c)}{c+h-c}$

Since f is cont's on (a,b) and differentiable on $(a,b)\setminus\{c\}$

So f is cont's on $[c,c+h]$ or $[c+h,c]$

diff on $(c,c+h)$ or $(c+h,c)$

So we can apply MVT to f on interval $[c,c+h]$ or $[c+h,c]$

then $\frac{f(c+h)-f(c)}{c+h-c}=f'(\xi_h)$ where ξ_h is between c and $c+h$

as $h\to 0$, $c+h\to c$, so $\xi_h\to c$

So $\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}=\lim_{h\to 0}f'(\xi_h)=\lim_{\xi_h\to c}f'(\xi_h)$ exists

□

Eg. f differentiable for $x>0$. and $f'(x) \to 0$ as $x \to +\infty$

let $g(x) = f(x+1) - f(x)$

Prove $g(x) \to 0$ as $x \to +\infty$

Pf: $$g(x) = \frac{f(x+1) - f(x)}{1} = \frac{f(x+1) - f(x)}{x+1-x}$$

Because f is diff for $x>0$, $\Rightarrow$ cont's for $x>0$

for any $x>0$, f is always cont's on $[x, x+1]$

diff on $(x, x+1)$

Apply MVT $g(x) = f'(\xi_x)$, $\xi_x \in (x, x+1)$

let $x \to +\infty$, $\xi_x > x$, $\Rightarrow \xi_x \to +\infty$

So $$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} f'(\xi_x)$$

$$= \lim_{\xi_x \to +\infty} f'(\xi_x) = 0$$

$\square$

* L'Hopital's Rule:

if $\lim_{x \to a} \frac{f(x)}{g(x)}$ is of form $\frac{0}{0}$, or $\frac{\infty}{\infty}$ (undetermined)

and $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ exists, then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Eg: $\lim_{x \to 0} \frac{\sin^2 x}{1 - \cos x}$, $\frac{\lim_{x \to 0} \sin^2 x = 0}{\lim_{x \to 0} 1 - \cos x = 0} = \frac{0}{0}$

$= \lim_{x \to 0} \frac{(\sin^2 x)'}{(1 - \cos x)'}$

Use L'Hopital's Rule.

$= \lim_{x \to 0} \frac{2 \sin x \cos x}{\sin x}$

$= \lim_{x \to 0} 2 \cos x = 2$

Ex. (1) $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$

$$\boxed{\lim_{x\to a}\left(f(x)+g(x)\right) \neq \lim_{x\to a}\left(f'(x)+g'(x)\right)}$$

(2) $c_0, c_1, \cdots c_n$ are real numbers satisfy

$$c_0+\frac{c_1}{2}+\cdots+\frac{c_n}{n+1}=0$$

Prove that $\underbrace{c_0+c_1x+\cdots+c_nx^n}_{f'(x)}=0$ has a sol'n between 0 and 1.

(Hint : consider $f(x)=c_0x+\frac{c_1}{2}x^2+\cdots+\frac{c_n}{n+1}x^{n+1}$)

Sol'n (1)

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)}$$

$$=\lim_{x\to 0}\frac{e^x-1}{e^x-1+xe^x}$$

$$=\lim_{x\to 0}\frac{e^x}{2e^x+xe^x}=\frac{1}{2}$$

(2) Consider $f(x)=c_0x+c_1\frac{x^2}{2}+\cdots\frac{c_n}{n+1}x^{n+1}$

note $f(0)=0,\quad f(1)=c_0+\frac{c_1}{2}+\cdots+\frac{c_n}{n+1}=0$

$$f'(x)=c_0+c_1x+\cdots+c_nx^n$$

Since f is cont's on $[0,1]$ and diff on $(0,1)$

apply MVT we have

$$0=\frac{f(1)-f(0)}{1-0}=f'(\xi) \quad \text{for some } \xi\in(0,1)$$

i.e. ξ is a sol'n to $c_0+c_1x+\cdots+c_nx^n=0$